Forced plumes

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This paper describes an investigation of the turbulent forced plumes generated by steady release of mass, momentum and buoyancy from a source situated in an extensive region of uniform or stably stratified fluid. The treatment, which is an extension of earlier work on buoyant plumes, also brings out the relationship between the *jet* and the *plume* as special cases of forced plumes.

The analysis shows that the behaviour of a forced plume from a source of finite size which delivers buoyancy, mass and momentum can in a uniform environment be related to that from a virtual point source of buoyancy and momentum only, and a treatment is given for the latter type of forced plume. When the environment is weakly stratified it is inferred that forced plumes can be related to point sources in the same way. In a uniform environment the plume fluid rises indefinitely; but when the environment is stably stratified, increasing the release of mass and momentum from a given source of buoyancy has the effect at first of reducing the total height of the plume, and only for very large flux of momentum does the height increase again, without limit. A description is given for the behaviour of vertical jets in a stably stratified environment, and for forced plumes of fluid with negative buoyancy.

Introduction

The plumes generated from steady sources of buoyancy in a uniform or stratified environment have been described by several authors. Laminar thermal plumes in a uniform environment were analysed by Gutman (1949) and Yih (1951, 1952), and Yih has also shown experimentally that such laminar flow is unstable quite close to the source (in terms of a local Rayleigh number) for all but the weakest sources. Thus the flow in most plumes will be effectively turbulent throughout the ascent, and this treatment will be restricted to turbulent flow.

Turbulent plumes in a uniform environment were first investigated by Schmidt (1941) using mixture length theories, and related experiments have been reported since by Yih (1951), Rouse, Yih & Humphreys (1952) and Railston (1954). In these experiments the nearest approximation to a pure source of buoyancy was Schmidt's electrically heated grid set in a circular hole in a table-top; Railston used a similar source near the bottom of a vertical chimney, so that he was actually considering constrained plumes in an induced current; and Rouse *et al.* (1952) carried out a careful set of experiments on free plumes above gas flames. Two treatments have been given for turbulent plumes in an ambient fluid in which the density varies linearly with height: Priestley & Ball (1955) were concerned specially with plumes rising from heated surfaces, while Morton, Taylor & Turner (1956) investigated the plumes from a virtual source of buoyancy only, and related these to experiments in which a steady stream of light fluid issued from a nozzle.

The purpose of this paper is to find the general effect of variations in the character of the source on the buoyant plumes which are produced. The plume generated from a source of finite size which delivers a flux of buoyancy, momentum and mass will be termed a *forced plume*. It will be shown that there is always an equivalent point (or virtual) source of buoyancy and momentum only which produces the same flow as the extended source, and the plumes from such point sources will also be called forced plumes. It may be noted that there can be no mass flow from a point source when the momentum flow is finite; for the momentum flow is of order (plume radius)^2 $\times$ (plume velocity)^2, so that the plume velocity near such a point source is O[(plume radius)$^{-1}$], and the mass flow is of order (plume radius)^2 $\times$ (plume velocity) which tends to zero at the source. The forcing considered here is from the source and not from the environment, which is assumed to be at rest except for the influence of the plume. Two limiting cases of the forced plume are the *jet* from a point source of momentum, and the pure *plume* from a source of buoyancy only.

In order that the problem will be mathematically tractable, certain assumptions must be made about the nature of the turbulent flow and its effect on fluid mixture across the mean boundaries of the plume. First, it will be assumed that well-developed turbulent flow in forced plumes is independent of viscosity (i.e. that local values of a Reynolds number are large), and of thermal conductivity (so that behaviour does not depend on the Prandtl number). Then, with the assumption of similarity of profiles at different cross-sections of the plume, it follows that the structure of turbulence within the plume and the rate of entrainment at its mean edge can depend only on the differences in mean density and mean vertical velocity between the plume axis and the ambient fluid. The treatment will be restricted to vertical plumes to eliminate the dependence on Richardson number of turbulent mixing of fluids with different densities across a horizontal interface. And finally it will be assumed that local density variations are everywhere so small in relation to some reference density that the inertia of unit volume of the fluid can be regarded as uniform. This eliminates the effect of density differences on mixing which may be important when large differences in temperature are maintained between the plume and ambient fluid (e.g. in turbulent flames), and is unlikely to introduce serious error for normal plumes, since any large density differences near the source are rapidly reduced as the fluid rises. It is then reasonable to assume that turbulence of similar character will be found in all forced plumes of the kind considered.

The general behaviour of forced plumes will be investigated now by an extension of the methods used by Morton *et al.* (1956) (hereafter referred to as paper I). As in their treatment, the analysis will refer to incompressible fluids, although it can be extended to include convection in the atmosphere by substituting potential temperatures and densities for ordinary temperatures and densities.

The model for forced plumes

The model used is very similar to that of paper I, being based on the assumptions: (i) that the ratio of the mean speed of inflow at the edge of a forced plume to the mean vertical speed on the plume axis is a constant, α, and (ii) that profiles of mean vertical velocity and buoyancy are each of similar form at all heights.

A feature of forced plumes which was ignored in paper I is the greater lateral spread of heat than of vertical momentum. Rouse *et al.* (1952) have illustrated this by experiments with buoyant plumes above isolated gas flames. They plotted their results in non-dimensional form, and chose the Gaussian profile

$$\exp(-96R^2/X^2)$$

as giving the best fit for a wide range of velocity measurements, and

$$\exp(-71R^2/X^2)$$

for the measurements of temperature excess; X is the height above the source and R the radial distance from the plume axis.

The entrainment constant α is clearly associated with the velocity profile rather than the temperature profile, and where the transport of material properties by the plume is of principal importance the temperature profile plays a secondary role even when buoyancy forces give rise to the flow. Thus it will be appropriate to seek a similarity solution for which α measures the rate of flow into a forced plume, with velocity profile characterized by the horizontal length-scale b, and with an associated buoyancy profile of the same shape but with a length-scale λb. λ and α are universal constants for forced plumes in which local density variations are small; they must be evaluated experimentally.

Forced plumes in a uniform environment

Consider an axially symmetrical plume generated from a roughly circular horizontal source in an incompressible environment. The plume will be assumed to have Gaussian profiles of mean vertical velocity $u(X, R) = U(X)\exp(-R^2/b^2)$ and mean buoyancy $g(\rho_e - \rho)(X, R) = \rho_0 P(X)\exp(-R^2/\lambda^2 b^2)$, with characteristic length-scales $b(X)$ and $\lambda b(X)$; $\rho_e(X)$ is the density in the environment and $\rho_0 = \rho_e(0)$ is the reference density. The equations representing conservation of mass, momentum and density deficiency (analogous to equations (6) of paper I) are, for a uniform environment, approximately

$$\left.\begin{aligned} \frac{d}{dX}(b^2U) &= 2\alpha bU, \\ \frac{d}{dX}(b^2U^2) &= 2\lambda^2 b^2 P, \\ \frac{d}{dX}(\lambda^2 b^2 UP) &= 0. \end{aligned}\right\} \tag{1}$$

Under the transformations $V = 2^{-\frac{1}{2}}bU$, $W = b^2U$ and $F = \lambda^2 b^2 UP/(1+\lambda^2)$, these reduce to

$$\frac{dW}{dX} = 2^{\frac{3}{2}}\alpha V, \quad \frac{dV^4}{dX} = (1+\lambda^2)FW, \quad \frac{dF}{dX} = 0. \tag{2}$$

$\pi\rho V^2$ is the momentum flux, $\pi\rho W$ the mass flux and $\pi\rho_0 F$ the buoyancy flux across a plane at position X in the forced plume.

The strength of a source will be defined by specifying the rates of discharge of buoyancy ($\pi\rho_0 F_0$), momentum ($\pi\rho V_0^2$) and mass ($\pi\rho W_0$), and will be labelled (F_0, V_0, W_0).

The forced plume from a source $(F_0, V_0, 0)$

It will be convenient to find the effect of discharge of momentum from a virtual point source of buoyancy before dealing with real sources of finite size. If there is a finite flux of buoyancy $\propto [b^2UP]_{X=0}$ and of momentum $\propto [b^2U^2]_{X=0}$ from a point source, both U and P are $O(1/b)$ at the source, and hence the corresponding mass flow $\propto [b^2U]_{X=0}$ must be zero. The plume is characterized by the parameters F_0 and V_0, and it may be verified that the transformations

$$\left.\begin{aligned} F = F_0, \quad V = |V_0|v, \quad W &= 2^{\frac{3}{4}}\alpha^{\frac{1}{2}}(1+\lambda^2)^{-\frac{1}{2}}|V_0|^{\frac{5}{2}}|F_0|^{-\frac{1}{2}}w, \\ X &= 2^{-\frac{3}{4}}\alpha^{-\frac{1}{2}}(1+\lambda^2)^{-\frac{1}{2}}|V_0|^{\frac{3}{2}}|F_0|^{-\frac{1}{2}}x, \end{aligned}\right\} \tag{3}$$

reduce equations (2) to the non-dimensional form

$$\frac{dw}{dx} = v, \quad \frac{dv^4}{dx} = w \operatorname{sgn} F_0, \tag{4}$$

where $\operatorname{sgn} F_0 = \pm 1$ according as $F \gtrless 0$. The reduced boundary conditions are

$$w = 0, \quad v = \operatorname{sgn} V_0, \quad \text{at } x = 0;$$

the buoyancy flux remains constant at all heights in a uniform environment.

When x is eliminated from equations (4)

$$w = 2^{\frac{3}{2}}5^{-\frac{1}{2}} \operatorname{sgn} v\, |v^5 - \operatorname{sgn} V_0|^{\frac{1}{2}}; \tag{5}$$

and when w is eliminated from equations (4), after integration

$$x = 2^{\frac{1}{2}}5^{\frac{1}{2}} \operatorname{sgn} F_0 \int_{\operatorname{sgn} V_0}^{v} |v_1^5 - \operatorname{sgn} V_0|^{-\frac{1}{2}} v_1^3\, dv_1. \tag{6}$$

Equations (5) and (6) provide a parametric solution for the forced plume in a uniform environment from the source $(F_0, V_0, 0)$ in terms of v. The following cases may be of interest:

(i) When the fluid delivered by the source is lighter than its environment and has upward momentum ($F_0 > 0$ and $V_0 > 0$), v increases steadily from its initial value $+1$ and

$$w = 2^{\frac{3}{2}}5^{-\frac{1}{2}}(v^5 - 1)^{\frac{1}{2}}, \quad x = 2^{\frac{1}{2}}5^{\frac{1}{2}}\int_1^v (v_1^5 - 1)^{-\frac{1}{2}} v_1^3\, dv_1. \tag{7a}$$

This solution can be written to an accuracy better than 1 % for $v \geqslant 2$

$$w = 2^{\frac{3}{2}}5^{-\frac{1}{2}}v^{\frac{5}{2}}, \quad x = 2^{\frac{3}{2}}5^{\frac{1}{2}}3^{-1}v^{\frac{3}{2}} - 1{\cdot}057; \tag{7b}$$

hence, for $x \geqslant 5$, $v \propto (x + 1{\cdot}057)^{\frac{2}{3}}$ and $w \propto (x + 1{\cdot}057)^{\frac{5}{3}}$, which is precisely the behaviour exhibited by the plume from a source $(F_0, 0, 0)$ of buoyancy only at $x = -1{\cdot}057$, below the given source. Figure 1 shows curves for the non-dimensional horizontal length-scale or effective plume 'radius' w/v ($\propto b$) and the non-dimensional vertical velocity on the axis v^2/w ($\propto U$). For purposes of comparison the corresponding curves for the 'radii' of plumes from virtual point sources of buoyancy only (marked 'plume') and momentum only (marked 'jet') are also shown by broken lines.

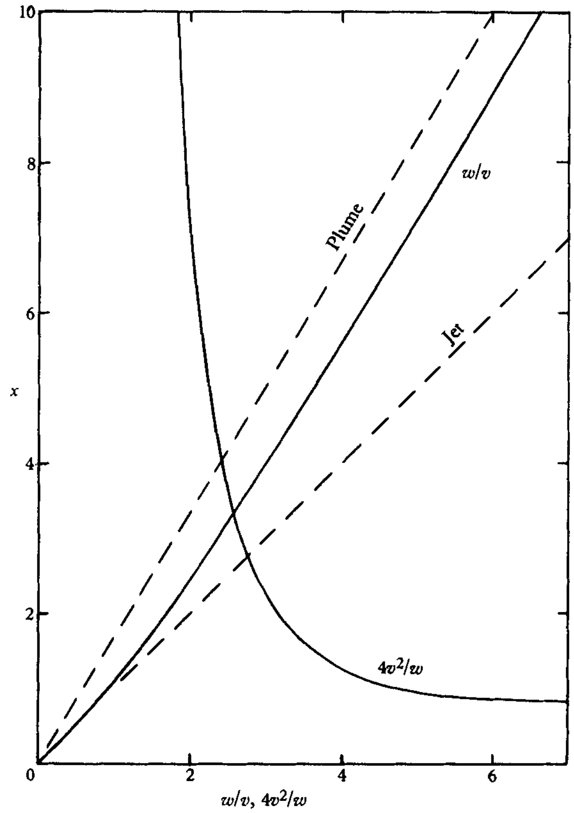

FIGURE 1. The behaviour of a forced plume from a virtual source of buoyancy and momentum, situated in a uniform environment. The non-dimensional quantities w/v, which is proportional to the horizontal length-scale or plume 'radius', and v^2/w, which is proportional to the vertical velocity on the axis, are plotted against x which is proportional to the height above the virtual source; the corresponding 'radii' for simple plumes and jets are shown in broken lines.

(ii) When the fluid from the source is heavier than its environment and has upward momentum ($F_0 < 0, V_0 > 0$), v decreases steadily from its initial value $+1$ to 0, and

$$w = 2^{\frac{3}{2}}5^{-\frac{1}{2}}(1-v^5)^{\frac{1}{2}}, \quad x = 2^{\frac{1}{2}}5^{\frac{1}{2}}\int_v^1 (1-v_1^5)^{-\frac{1}{2}}\, v_1^3\, dv_1.* \tag{8}$$

The fluid in the forced plume will rise to a maximum height

$$x = 2^{\frac{1}{2}}5^{\frac{1}{2}}\int_0^1 (1-v_1^5)^{-\frac{1}{2}}\, v_1^3\, dv_1 = 1{\cdot}454,$$

* Note that x can also be written $2^{\frac{1}{2}}5^{\frac{1}{2}}\{\beta(\frac{4}{5}, \frac{1}{2}) - \beta(\frac{4}{5}, \frac{1}{2}; v^5)\}$, where $\beta(\frac{4}{5}, \frac{1}{2})$ is a β-function and $\beta(\frac{4}{5}, \frac{1}{2}; v^5)$ an incomplete β-function of argument v^5.

and will then spread sideways as it falls back; some plume fluid will remain near the highest point because of entrainment from above (which has been neglected). The present method can provide a solution only for the ascent. The solution curves for $w/v\,(\propto b)$, $v^2/w\,(\propto U)$ and $-1/w\,(\propto P)$ against the dimensionless height x are plotted in figure 2.

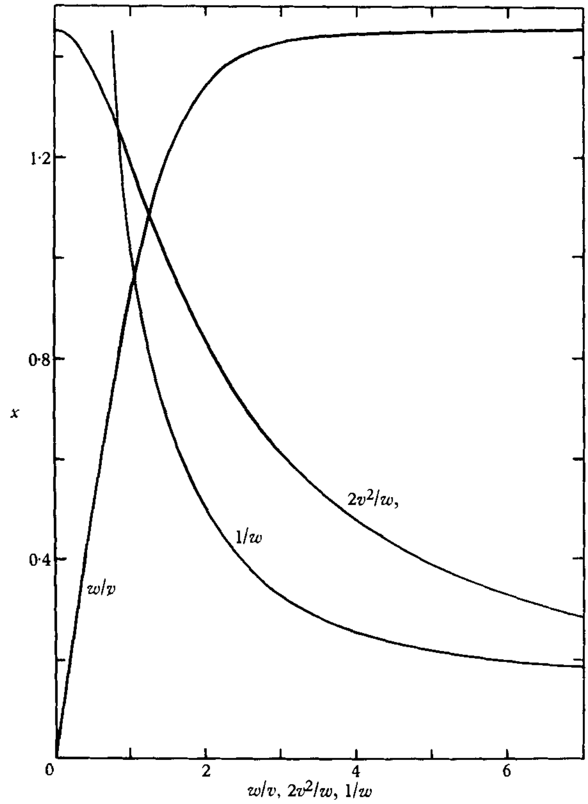

FIGURE 2. The behaviour of a forced plume from a virtual source of momentum and negative buoyancy situated in a uniform environment. w/v is proportional to the plume 'radius', v^2/w to the vertical velocity at the axis, and $1/w$ to the deficiency of temperature at the axis of the plume relative to that of the environment.

(iii) When the fluid from the source is lighter than the environment but has downward momentum ($F_0 > 0$, $V_0 < 0$), v increases steadily from its initial value -1, and

$$w = 2^{\frac{3}{2}}5^{-\frac{1}{2}}\operatorname{sgn} v(1+v^5)^{\frac{1}{2}}, \quad x = 2^{\frac{1}{2}}5^{\frac{1}{2}}\int_{-1}^{v}(1+v_1^5)^{-\frac{1}{2}}v_1^3\,dv_1. \tag{9a}$$

The forced plume descends to a 'singular point' at

$$x = 2^{\frac{1}{2}}5^{\frac{1}{2}}\int_{-1}^{0}(1+v_1^5)^{-\frac{1}{2}}v_1^3\,dv_1 = -1{\cdot}454$$

below the virtual source, and then rises again. The solution for a descending forced plume is valid physically only until the flow spreads sideways, and is just case (ii) inverted. The ascending part of the solution (for $v > 0$) describes a forced plume from an actual source where the mass flow is too large for case (i) to apply.

Approximate expressions for $v \geqslant 3$ are

$$w = 2^{\frac{3}{2}}5^{-\frac{1}{2}}v^{\frac{5}{2}}, \quad x = 2^{\frac{3}{2}}5^{\frac{1}{2}}3^{-1}v^{\frac{3}{2}} - 3{\cdot}253; \tag{9b}$$

hence for $x \geqslant 8$ the forced plume behaves closely as though it were generated from a virtual source of buoyancy at $x = -3{\cdot}253$, below the given source. There is now an initial region of accelerated flow above the source.

The forced plume from a source (F_0, V_0, W_0)

The effect on a forced plume of independent variation of the mass flux W_0 from the source can be found only by considering the effects due to a source of finite area. The behaviour of the plume is represented again by equations (4). For an extensive source it may be assumed that the flow is directed upwards from the source (i.e. $\operatorname{sgn} V_0 = \operatorname{sgn} W_0 = 1$), so that the boundary conditions at the source (F_0, V_0, W_0) are

$$v = 1, \quad w = 2^{\frac{3}{2}}5^{-\frac{1}{2}}|\Gamma|^{\frac{1}{2}} \quad \text{at } x = 0, \tag{10}$$

where $\Gamma = 2^{\frac{5}{8}}5\alpha^{-1}(1+\lambda^2)F_0V_0^{-5}W_0^{\frac{2}{3}}$ is a dimensionless parameter representative of the forced plume in a uniform environment.

A plume which is the same above the level $x = 0$ can be generated from a virtual point source $(F_0, \gamma V_0, 0)$ situated at a certain height $x = -\bar{x}$. (Recall that the flux of buoyancy is constant in a uniform environment.) This equivalent plume must satisfy the same non-dimensional equations as the forced plume from the source (F_0, V_0, W_0), and the modified boundary conditions $v' = \gamma$ and $w' = 0$ at $x' = 0$, where the dashes are used to refer symbols to the equivalent plume from $(F_0, \gamma V_0, 0)$; the corresponding solution is

$$\left.\begin{aligned} v' \geqslant \gamma, \quad w' &= 2^{\frac{3}{2}}5^{-\frac{1}{2}}|v'^5 - \gamma^5|^{\frac{1}{2}}, \\ x' &= 2^{\frac{1}{2}}5^{\frac{1}{2}} \operatorname{sgn} F_0 \int_{\gamma}^{v'} |v_1'^5 - \gamma^5|^{-\frac{1}{2}} v_1'^3\, dv_1'. \end{aligned}\right\} \tag{11}$$

The two plumes will be identical above the level of the actual source if at $x' = \bar{x}$, $v' = 1$ and $w' = 2^{\frac{3}{2}}5^{-\frac{1}{2}}|\Gamma|^{\frac{1}{2}}$; these conditions determine γ and $\bar{x}$ as

$$\left.\begin{aligned} \gamma^5 &= 1 - \Gamma, \\ \bar{x} &= 2^{\frac{1}{2}}5^{\frac{1}{2}} \operatorname{sgn} F_0 \int_{\gamma}^{1} |v_1'^5 - \gamma^5|^{-\frac{1}{2}} v_1'^3\, dv_1' \\ &= 2^{\frac{1}{2}}5^{\frac{1}{2}}|\gamma|^{\frac{3}{2}} \operatorname{sgn} F_0 \int_{\operatorname{sgn}\gamma}^{1/|\gamma|} |t^5 - \operatorname{sgn}\gamma|^{-\frac{1}{2}} t^3\, dt, \end{aligned}\right\} \tag{12}$$

where $v_1' = |\gamma|\, t$. The significance of these solutions depends on the parameters of the physical system in a complicated way; it will be investigated further in a separate paper on convection by forced plumes in the atmosphere. The following comments may be of interest here:

(*a*) when $0 < \Gamma < 1$, forced plumes behave as though from a virtual source $(F_0, \gamma^{\frac{1}{3}}V_0, 0)$ situated at

$$x = -2^{\frac{1}{2}}5^{\frac{1}{2}}\gamma^{\frac{3}{2}} \int_1^{1/\gamma} (t^5 - 1)^{-\frac{1}{2}} t^3\, dt.$$

The full solution is that of case (i) for the forced plume $(F_0, V_0, 0)$ given above;

(*b*) when $\Gamma = 1$, forced plumes behave as the plume $(F_0, 0, 0)$ from a source of buoyancy only at $x = -2{\cdot}108$;

(*c*) when $\Gamma > 1$, forced plumes behave as though from a virtual source $(F_0, -|\gamma| V_0, 0)$ situated at

$$x = 3{\cdot}162\,|\gamma|^{\frac{3}{5}} \int_{-1}^{1/|\gamma|} (t^5+1)^{-\frac{1}{2}} t^3\, dt;$$

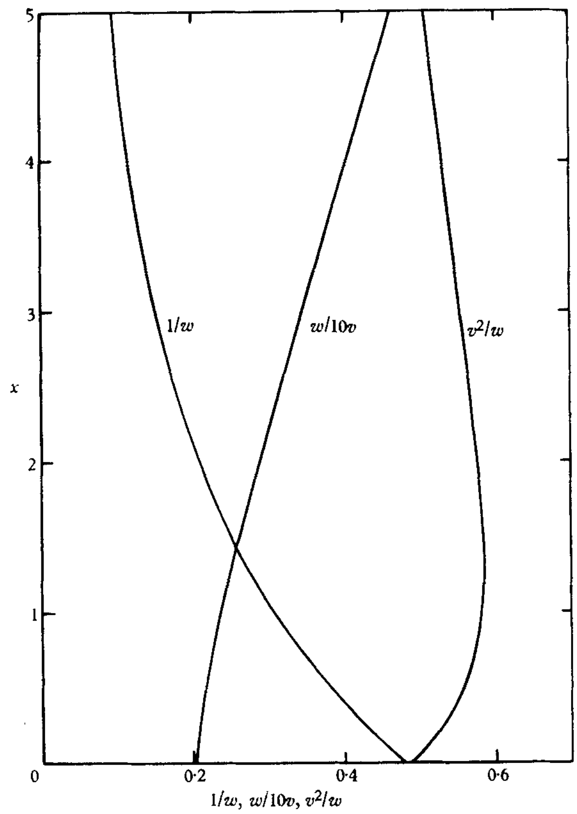

FIGURE 3. The behaviour of a forced plume from an extensive source discharging heated fluid and situated in a uniform environment; for this source $\Gamma = 2{\cdot}69$. w/v is proportional to the plume 'radius', v^2/w to the vertical velocity at the axis, and $1/w$ to the temperature excess at the axis over that in the environment.

this integral can be positive, in which case the virtual source lies *above* the actual source (see case (iii) above). A particular example has been worked for $\Gamma = 2{\cdot}69$ and the results are shown in figure 3. The curves show that when $\Gamma > 1$ the plume fluid is at first accelerated above the source, but that the vertical velocity ultimately decreases steadily with height and the plume becomes straight-sided;

(*d*) when $\Gamma < 0$ the forced plume behaves as though from a virtual source $(F_0, \gamma^{\frac{1}{2}} V_0, 0)$ at

$$x = -3{\cdot}162\gamma^{\frac{3}{5}} \int_{1/\gamma}^{1} (1-t^5)^{-\frac{1}{2}} t^3\, dt$$

(see case (iii) above).

The behaviour of a forced plume in a uniform environment is characterized by the dimensionless parameter Γ, and relative to the straight sided plume from a virtual source of buoyancy the entrainment near the source is increased when $\Gamma < 1$ and decreased when $\Gamma > 1$. Hence the most rapid removal of plume fluid from the neighbourhood of the source is obtained for small V_0^5 and large $W_0^2 F_0$, by releasing the fluid slowly from a large aperture and giving it the maximum buoyancy. The most rapid mixing of the effluent with its environment is obtained in the jet.

Values for the constants λ and α

For *Gaussian* profiles, a value $\lambda = 1{\cdot}16$ can be obtained in the case of thermal diffusion from the profiles recommended by Rouse *et al.* (1952). An improved value $\alpha = 0{\cdot}082$ for the entrainment constant has been found by comparison of the predictions of this analysis with experimental results for jets and plumes (see Morton 1959).

Forced plumes in a stably stratified environment

The foregoing treatment has shown that modifications to the shape of forced plumes caused by changes in the source (i.e. in F_0, V_0 and W_0) are restricted mainly to the lower parts of the plumes. In a weakly stratified environment the effects of stratification will be relatively small near the source, but will dominate the upper regions of plumes. Indeed, the environment can be quite strongly stratified and still produce a negligible effect on the lower part of the forced plume (for example, between the virtual point source and the actual source). This may be demonstrated very clearly by comparing the widths of pure plumes from point sources of buoyancy in uniform and stably stratified environments (see paper I); stratification leads to $< 1\,\%$ increase in width at one-third of the plume height, $3\,\%$ at half the height and $< 8\,\%$ at two-thirds of the height. Similarly, for vertical jets (see below), stable stratification causes $< 2\,\%$ increase in width half-way up the jet, and $5\,\%$ increase at two-thirds of the jet height. Consequently, in a stratified environment the shape of forced plumes in their lower parts will be similar to that which has been predicted for a uniform environment, and in their upper parts will be like that described in paper I. Thus, the calculations of the previous section for the position of virtual point sources in a uniform environment can be carried over directly to the case of a stratified environment with very great saving of labour and without appreciable increase of error; but further analysis is needed to find the dependence on F_0, V_0 and W_0 of the height of the plume top, since this will depend critically on stratification.

For a stratified environment the use of Gaussian profiles is unsatisfactory (Morton 1959) so they will be replaced for this section by 'top hat' profiles, with the mean vertical velocity constant across a section of width $2b(X)$ and zero outside it and the mean buoyancy constant across a symmetrically situated profile of width $2\lambda b(X)$ and zero outside. This change of profile does not affect the height calculated for the plume top, but it does modify the values of the constants; for the plume with 'top hat' profiles, $\alpha = 0{\cdot}116$ and $\lambda = 1{\cdot}108$.

The general forced plume from the source (F_0, V_0, W_0) is in a uniform environment equivalent to a related plume from a virtual source of buoyancy and momentum, and so it will be sufficient here to consider forced plumes (F_0, V_0, 0) from virtual sources. Only the case of stable stratification with a uniform density gradient will be investigated. The physical conditions of the problem provide the three parameters F_0, V_0 and $G = (g/\rho_0)(d\rho_e/dX)$ (a measure of the stratification); one dimensionless parameter, GV_0^4/F_0^2, can be formed from these, and any two form a basis for a reduction of the system to non-dimensional form.

The equations for plumes with 'top hat' profiles (corresponding with equation (2) for Gaussian profiles) are

$$\frac{dW}{dX} = 2\alpha V, \quad \frac{dV^4}{dX} = 2\lambda^2 FW, \quad \frac{dF}{dX} = -GW, \tag{13}$$

where in this case $V = bU$, $W = b^2U$, $F = b^2UP$ in order to preserve the physical meaning of V, W and F. The transformations

$$F = |F_0| f, \quad V = 2^{\frac{1}{4}}\lambda^{\frac{1}{2}}|F_0|^{\frac{1}{2}}G^{-\frac{1}{4}}v, \quad W = 2^{\frac{5}{8}}\alpha^{\frac{1}{2}}\lambda^{\frac{1}{4}}|F_0|^{\frac{3}{4}}G^{-\frac{5}{8}}w,$$
$$X = 2^{-\frac{5}{8}}\alpha^{-\frac{1}{2}}\lambda^{-\frac{1}{4}}|F_0|^{\frac{1}{4}}G^{-\frac{3}{8}}x,$$

reduce these equations to their simplest non-dimensional form,

$$\frac{dw}{dx} = v, \quad \frac{dv^4}{dx} = fw, \quad \frac{df}{dx} = -w; \tag{14}$$

and the corresponding boundary conditions at $x = 0$ are,

$$w = 0, \quad v = 2^{-\frac{1}{4}}\lambda^{-\frac{1}{2}}|F_0|^{-\frac{1}{2}}G^{\frac{1}{4}}V_0 = v_0 \text{ (say)}, \quad f = \operatorname{sgn} F_0. \tag{15}$$

The following cases include all plumes emitted upwards from sources of finite area.

(i) $F_0 > 0$ and $V_0 > 0$, corresponding with actual sources for which $0 < \Gamma < 1$. From the last two of equations (14)

$$2v^4 + f^2 = 2v_0^4 + 1 = \frac{1}{1-\sigma},$$

where $\sigma = GV_0^4/(\lambda^2F_0^2 + GV_0^4)$ is the most convenient form for the representative parameter. Define the new independent variable $s = 2(1-\sigma)v^4$; then

$$f = \pm\left(\frac{1-s}{1-\sigma}\right)^{\frac{1}{2}}, \tag{16}$$

where the positive sign is taken where the buoyancy force acts upwards and the negative sign where it acts downwards. The value of v increases from $v = v_0$ at the source to a maximum $v = (v_0^4 + \frac{1}{2})^{\frac{1}{4}}$ at the level where f vanishes, and then decreases steadily to zero at the top of the plume ($v^2 \propto$ momentum flux); the corresponding values for s are $s = \sigma\,(< 1)$ at the source, $s = 1$ where $f = 0$, and $s = 0$ at the plume top. f decreases steadily from $f = 1$ at the source to $f = -(1-\sigma)^{-\frac{1}{2}}$ at the plume-top.

From the first two of equations (14), for the lower part of the plume ($\sigma \leqslant s \leqslant 1$),

$$\begin{aligned} w^2 &= 2^{-\frac{1}{4}}(1-\sigma)^{-\frac{3}{4}}\int_\sigma^s t^{\frac{1}{4}}(1-t)^{-\frac{1}{2}}\,dt \\ &= 2^{-\frac{1}{4}}(1-\sigma)^{-\frac{3}{4}}\{\beta(\tfrac{5}{4}, \tfrac{1}{2}; s) - \beta(\tfrac{5}{4}, \tfrac{1}{2}; \sigma)\}, \end{aligned} \tag{17a}$$

and in the upper part of the plume ($1 \geqslant s \geqslant 0$)

$$w^2 = 2^{-\frac{1}{4}}(1-\sigma)^{-\frac{3}{4}}\{2\beta(\tfrac{5}{4}, \tfrac{1}{2}) - \beta(\tfrac{5}{4}, \tfrac{1}{2}; s) - \beta(\tfrac{5}{4}, \tfrac{1}{2}; \sigma)\}. \tag{17b}$$

The parametric solution is now completed by finding x from equation (14, ii); for the present purpose we need only the height of the plume-top, x_{max}, which is the greatest height to which fluid emitted from the source can penetrate

$$x_{max} = 2^{-\frac{7}{8}}(1-\sigma)^{-\frac{1}{8}}\left\{\int_\sigma^1 \frac{dt}{\sqrt{(1-t)}\sqrt{\{\beta(\frac{5}{4}, \frac{1}{2}; t) - \beta(\frac{5}{4}, \frac{1}{2}; \sigma)\}}} + \int_0^1 \frac{dt}{\sqrt{(1-t)}\sqrt{\{2\beta(\frac{5}{4}, \frac{1}{2}) - \beta(\frac{5}{4}, \frac{1}{2}; t) - \beta(\frac{5}{4}, \frac{1}{2}; \sigma)\}}}\right\}. \tag{18}$$

A graph of x_{max} plotted against σ is shown in figure 4; the curve for expression (18) is labelled $0 < \Gamma < 1$. The limiting case $\sigma = 0$ for the plume from a virtual source of buoyancy only has a greatest height $x_{max} = 2{\cdot}805$, which agrees with the value

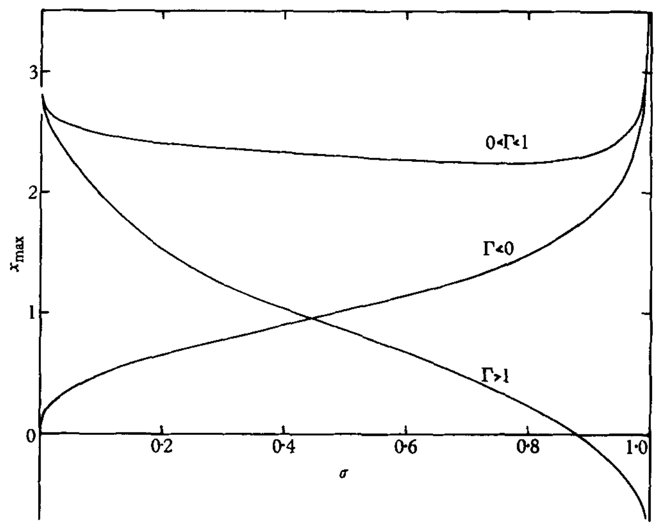

FIGURE 4. The non-dimensional height (x_{max}) of the top of a forced plume in a stably stratified environment, plotted as a function of the dimensionless parameter

$$\sigma = GV_0^4/(\lambda^2F_0^2 + GV_0^4).$$

The curves drawn are for the following cases: (i) $0 < \Gamma < 1$ represents the plume from a virtual source of positive buoyancy and upward momentum, (ii) $\Gamma < 0$ is that for a source of negative buoyancy and upward momentum, and (iii) $\Gamma > 1$ is that for a source of positive buoyancy and downward momentum.

2·80 +, given in paper I; the complete solution for the case $\sigma = 0$ can be found in that paper. As σ is increased the value x_{max} decreases, at first rapidly owing to the sudden increase in mixing near the source and then more slowly. When σ exceeds 0·8 the buoyancy flux from the source is becoming less important than that of momentum, and forced plumes can be projected to any height by sufficiently increasing the momentum of the fluid as it leaves the sources; the necessary increase may be very large, for example, $GV_0^4/\lambda^2F_0^2$ must be increased to about 100 before the height of the forced plume is again as great as that of the simple plume from a virtual point source of buoyancy. (Actual heights are directly proportional to dimensionless heights if V_0 only is varied.)

(ii) $F_0 < 0$ and $V_0 > 0$, corresponding with actual sources for which $\Gamma < 0$. The appropriate boundary conditions at $x = 0$ are

$$f = -1, \quad w = 0, \quad v = v_0 = 2^{-\frac{1}{4}}\sigma^{\frac{1}{4}}(1-\sigma)^{-\frac{1}{4}};$$

and the greatest height to which the heavy plume fluid rises is

$$x_{\max} = 2^{-\frac{7}{8}}(1-\sigma)^{-\frac{1}{8}} \int_0^{\sigma} \frac{dt}{\sqrt{(1-t)}\sqrt{\{\beta(\frac{5}{4}, \frac{1}{2}; \sigma) - \beta(\frac{5}{4}, \frac{1}{2}; t)\}}}. \tag{19}$$

The variation in height of the plume-top with σ is shown in figure 4 by the curve labelled $\Gamma < 0$. A relatively small flux of momentum is needed to lift the plume top well clear of the source, but thereafter the height increases slowly with increasing V_0. The parts of the curves near $\sigma = 1$ in figure 4 are very similar because here the forced plume is dominated by the momentum flux from the source.

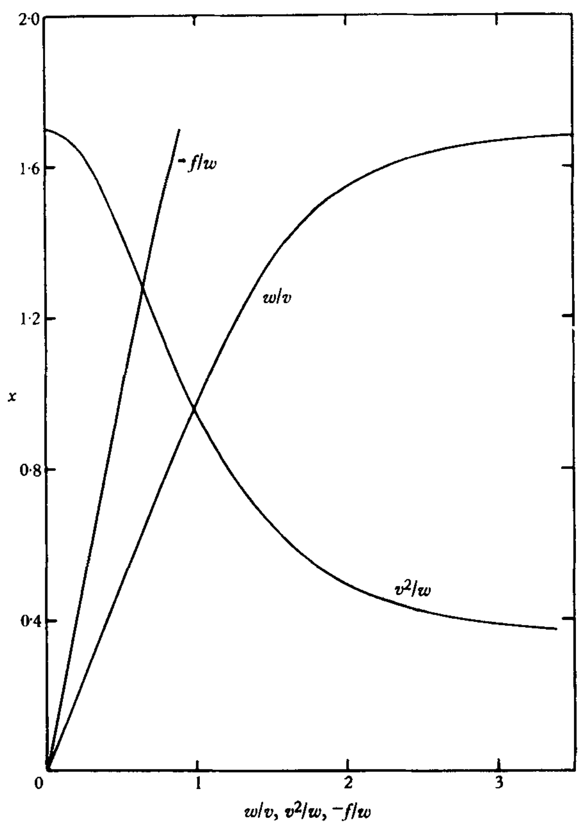

FIGURE 5. The behaviour of the jet from a virtual source of momentum, situated in a stably stratified environment. The non-dimensional quantities plotted are w/v, which is proportional to the horizontal length scale or 'radius' of the jet, v^2/w, proportional to the vertical velocity on the axis, and $-f/w$, proportional to the reduction of temperature at the axis below that of the ambient fluid at that level; x is the non-dimensional height above the virtual source.

(iii) $F_0 > 0$ and $V_0 < 0$, corresponding with actual sources for which $\Gamma > 1$. The initial downwards flow corresponds with case (ii) inverted; the height of the plume-top above the source in the subsequent ascent is

$$x_{\max} = 2^{-\frac{7}{8}}(1-\sigma)^{-\frac{1}{8}}\left\{-\int_0^\sigma \frac{dt}{\sqrt{(1-t)}\sqrt{\{\beta(\frac{5}{4},\frac{1}{2};\sigma)-\beta(\frac{5}{4},\frac{1}{2};t)\}}} + \int_0^1 \left[\frac{1}{\sqrt{\{\beta(\frac{5}{4},\frac{1}{2};\sigma)+\beta(\frac{5}{4},\frac{1}{2};t)\}}} + \frac{1}{\sqrt{\{2\beta(\frac{5}{4},\frac{1}{2})+\beta(\frac{5}{4},\frac{1}{2};\sigma)-\beta(\frac{5}{4},\frac{1}{2};t)\}}}\right]\frac{dt}{\sqrt{(1-t)}}\right\}. \tag{20}$$

The position of the plume top relative to the virtual source is shown by the curve labelled $\Gamma > 1$ in figure 4, and the lowest level to which the plume fluid penetrates can be found from the curve $\Gamma < 0$ in the same figure.

The vertical jet from a source $(0, V_0, 0)$ *in a stratified environment* (G)

In this case the scale of the motion is determined by V_0 and G; the transformations

$$F = 2^{-\frac{1}{2}}\lambda^{-1}V_0^2G^{\frac{1}{2}}f, \quad V = V_0v, \quad W = 2^{\frac{1}{4}}\alpha^{\frac{1}{2}}\lambda^{-\frac{1}{2}}V_0^{\frac{3}{2}}G^{-\frac{1}{4}}w,$$
$$X = 2^{-\frac{3}{4}}\alpha^{-\frac{1}{2}}\lambda^{-\frac{1}{2}}V_0^{\frac{1}{2}}G^{-\frac{1}{4}}x,$$

reduce the equations for the jet to the same form as equations (14), and the boundary conditions at $x = 0$ to $v = 1$, $w = 0$ and $f = 0$. The solution is

$$\left.\begin{aligned} v = s^{\frac{1}{4}}, \quad w = 2^{\frac{1}{4}}\{\beta(\tfrac{5}{4},\tfrac{1}{2})-\beta(\tfrac{5}{4},\tfrac{1}{2};s)\}^{\frac{1}{2}}, \\ f = -2^{\frac{1}{2}}(1-s)^{\frac{1}{2}}, \quad x = 2^{-\frac{3}{4}}\int_s^1 \frac{dt}{\sqrt{(1-t)}\sqrt{\{\beta(\frac{5}{4},\frac{1}{2})-\beta(\frac{5}{4},\frac{1}{2};t)\}}}; \end{aligned}\right\} \tag{21}$$

the height of the plume top is $x_{\max} = 1{\cdot}70$. Figure 5 shows these results; $v^2/w \propto$ vertical velocity within the plume, $w/v \propto$ radius of the plume, and $f/w \propto$ buoyancy of the plume fluid. These curves may be compared with those for a simple plume shown in figure 1 of paper I; the most obvious and important difference is the more rapid spreading of the jet.

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